Finite-temperature nonlinear dynamics in cavity QED: A Thermofield Dynamics Approach

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Heath-bath effects in the dynamics of atom + cavity system are studied. The temperature effects are explored using thermofield dynamics formalism. It is found that the dynamics of the system is sensitive to small changes in the temperature and the thermal effects lead to increasing instabilities by causing transitions from regular to chaotic motion.

I. INTRODUCTION

Cavity quantum electrodynamics deals with studying the interaction of atoms with photons in high-finesse cavities in a wide range of the electromagnetic spectrum, from microwaves to visible light. The fact that the system "atom + cavity mode" is a quantum system makes cavity quantum electrodynamics (QED) an excellent testing ground for such important issues of modern quantum physics as quantum measurement theory, entanglement, quantum computation, quantum interference and at the same time provides a unique possibility for trapping, cooling and manipulating of atoms [2].. Practical importance of cavity QED is mainly related to potential possibility for manipulating atoms and photons in mesoscopic scales. Therefore, in recent years cavity QED has become one of the hot topics both in theoretical and experimental context [1–3]. In particular, a number of new phenomena, such as the realization of quantum phase gate [4], creation of Fock states of the radiation field [5, 6] and quantum non-demolition measurements [2] have been found. The dynamics of a single atom trapped in a microcavity is governed by quantum electrodynamics. This makes cavity QED an interdisciplinary area as many subfield of physics, such as quantum and atomic optics, cold atom physics, physics of nanosized systems and quantum information, may use important results of the cavity QED.

Recently cavity QED is considered in dealing with nonlinear dynamics [7, 9]. Mapping quantum equations of motion onto classical ones, for the Jaynes-Cummings Hamiltonian, which includes recoil motion of the atom, Prants. et. al, explored phase-space dynamics of the atom interacting with a single cavity mode by analyzing Poincare surface sections and calculating Lyapunov exponents. In such a case classical and not quantum chaos described by classical equations of motion corresponding to Jaynes-Cummings hamiltonian are studied that are classical counterpart of the atom+photon+cavity system that are described by Prants et al [7]-[10]. This system can be considered as a periodically driven quantum system as most of the experiments on the study of chaos in atoms explore interaction of cold atoms with the standing waves of various type, such as phase-modulated standing wave [22], amplitude-modulated standing wave and pulsed standing wave [23].

In this paper we extend the approach developed by Prants et al. [7, 9] to the case of the system coupled to a heat bath i.e. at finite temperature. For this purpose we use a real-time finite-temperature quantum field theory, thermofield dynamics [13–15]. In particular, by solving temperature-dependent classical equations of motion we explore Poincare surfaces sections, Lyapunov exponents and Levy flights in finite-temperature cavity QED. The role of a thermal bath, noise and dissipation effects in cavity QED is of importance both from fundamental and practical viewpoints. This paper is organized as follows: in the next section we present brief description of TFD formalism. Section 3 briefly recalls the results for $T = 0$ obtained by Prants et al. Section 4 presents treatment of "atom + confining mode" system at finite temperature, while numerical results for Poincare surfaces of section, Levy flights and Lyapunov exponent at $T = 0$ are given in section 5. Finally, section 6 presents some concluding remarks.

II. THERMOFIELD DYNAMICS

TFD is a real time operator formalism of quantum field theory at finite temperature in which any physical system can be constructed from a temperature-dependent vacuum which is a pure state [13]-[20]. For finite temperature it has been recognized for a long time that the Hilbert space has to be doubled. This is achieved with Thermofield Dynamics. One of the objective was to get a theory that real time and at the same time it is at finite temperature. The second Hilbert space is introduced along with a set of operators that are similar to be distinct from the normal operators. The final results don't depend explicitly on this second set of operators. However, Bogoliubov transformation mixes the two set of operators.This brings in factors that are temperature-dependent.The vacuum is also temperature-dependent and the creation and annihilation operators are converted to temperature-dependent operators, such that the annihilation operator acting on this vacuum gives zero. For a long time it was thought that these operators are ghosts. However, Umezawa showed that these operators can be viewed as degrees of freedom of the heat bath in the classical theory [13]. Later on it was shown by Santana and Khanna that one set gives the set of observables and the other gives symmetry operators. This argument is based on group theory and is given in the ref. [20]

Thus TFD is a powerful tool for exploring quantum dynamics of a system at finite temperature provided its Hamiltonian can be represented in terms of annihilation and creation operators. It has found many applications in condensed matter physics [13, 16, 17], especially in superconductivity theory and related topics. Recently TFD has been applied to explore quantum chaos in the Yang-Mills-Higgs system [18] and for the calculation of the spectra of a strongly interacting bound system [19]. In this work we apply TFD prescription to Jaynes-Cummings model. It should be noted that TFD has been applied earlier to Jaynes-Cummings model [21] where the thermal noise effects in quantum optics are studied.

Applying TFD prescription to a quantum system implies performing two actions: [14, 15]
i) doubling of the Fock space which means that all operators are doubled by introducing their tilded partners describing the effects of a heath bath; and
ii) using Bogoliubov transformations, which makes Hamiltonian of the system temperature-dependent . With respect to a Hamiltonian operator written in terms of annihilation and creation operators, doubling means that the total Hamiltonian is written as

$$\hat{H} = H - \tilde{H}, \tag{1}$$

while the Bogoliubov transformations which are given by

$$a = a(\beta)\cosh\theta + \tilde{a}^{\dagger}(\beta)\sinh\theta$$

$$a^{\dagger} = a^{\dagger}(\beta)\cosh\theta + \tilde{a}(\beta)\sinh\theta$$

$$\tilde{a} = a^{\dagger}(\beta)\sinh\theta + \tilde{a}(\beta)\cosh\theta \tag{2}$$

$$\tilde{a}^\dagger = a(\beta)\sinh\theta + \tilde{a}^\dagger(\beta)\cosh\theta,$$

make this Hamiltonian temperature-dependent.

Here

$$\beta = \frac{\omega}{k_B T}; \quad \sinh^2\theta = (e^\beta - 1)^{-1} \tag{3}$$

and the annihilation and creation operators satisfy the following commutation relations:

$$[a(\beta), a^+(\beta)] = 1 \quad [\tilde{a}(\beta), \tilde{a}^+(\beta)] = 1 \tag{4}$$

All other commutation relations are zero.

We note that we are dealing with a system in equilibrium, i.e. the temperature, T, is constant. Since the Hamiltonian of the Jaynes-Cummings model is written in terms of annihilation and creation operators, it is convenient to use TFD formalism to treat this model at finite temperature. Applying the above prescription to the Jaynes-Cummings model we transform the Hamiltonian of the system into the temperature-dependent form that allows us to treat chaos using the same approach as that used by Prants et.al [9].

III. CAVITY QED AT $T = 0$

In this section we briefly recall the case of $T = 0$ which is recently explored in detail in a series of papers by Prants et.al. [7]-[12]. The simplest cavity QED system is a single two-level atom interacting with a single standing wave mode moving along the x-axis with the frequency ω_f.

The dynamics of this system in the presence of atomic recoil motion is described by Jaynes-Cummings Hamiltonian, which is written as

$$\hat{H} = \frac{\hat{P}^2}{2m} + \hbar\omega_a \hat{S}_z + \hbar\omega_f \hat{a^+}\hat{a} - \hbar\Omega_0(\hat{a^+}\hat{S_-} + \hat{a}\hat{S_+})\cos(k_f\hat{x}), \tag{5}$$

where $\hat{S}_z$, $\hat{S}_+$ and $\hat{S}_-$ - are expressed in terms of pauli matrices i.e. $\hat{S}_{+,-} = (\hat{S}_x, \hat{S}_y)$ and $\hat{S}_z$ is the z - component, and $\hat{a^+}$ and $\hat{a}$ are the creation and annihilation operators, respectively, describing a selected mode of the radiation field of the frequency ω_f and the wave number k_f in a lossless cavity. The parameter Ω_0 is the amplitude value of the atom-field dipole coupling and depends on the position of an atom inside a cavity. To treat the nonlinear dynamics of this system Prants et. al. derived first the quantum equations of motion for the operators external atomic operators, P and x and slowly varying amplitudes of the field and spin operators: $\hat{a}(t) = \hat{a}e^{-i\omega_f t}$, $\hat{a}^+(t) = \hat{a}^+ e^{i\omega_f t}$, $\hat{S}_\pm(t) = \hat{S}_\pm e^{\mp i\omega t}$ and $\hat{S}_z(t) = \hat{S}_z$.

Taking the averages of all operators over an initial quantum state(which is a product of the translational, electronic, and the radiation field states) the quantum equations of motions can be replaced by the equations for the expectation values of the operators [9]. The equations of motions for these averages are written as

$$\dot{x} = \alpha p$$

$$\dot{p} = -2(a_x s_x + a_y s_y) sinx$$

$$\dot{s}_x = -\delta s_y + 2a_y s_z cosx$$

$$\dot{s}_y = \delta s_x - 2a_x s_z \cos x \tag{6}$$

$$\dot{a}_x = -s_y \cos x$$

$$\dot{a}_y = s_x \cos x,$$

where the expectation values are defined as

$$x = k_f \langle \hat{x} \rangle, p = \langle \hat{p} \rangle / \hbar k_f$$

$$s_x = \langle s_- + s_+ \rangle /2, s_y = \langle s_- - s_+ \rangle /2i$$

$$a_x = \langle a + a^+ \rangle /2, a_y = \langle a - a^+ \rangle /2i, \tag{7}$$

$$\alpha = \hbar k_f^2 / m\Omega_o, \tau = \Omega_o t$$

In the ref. [9] the dynamics of the cavity+atom system at $T = 0$ is treated by solving Eq. (6) and analyzing the solutions in terms of Poincare surface sections and Lyapunov exponents. The extension of the zero temperature results [9] to non-zero temperature is done here by using the formalism of TFD.

IV. CAVITY QED AT NON-ZERO TEMPERATURE

Applying TFD prescription to the cavity QED Hamiltonian (5), we have

$$\hat{H} = \frac{P^2 - \tilde{P}^2}{2m} + \hbar\omega_f (a^+(\beta)a(\beta) - \tilde{a}^+(\beta)\tilde{a}(\beta))$$

$$+\hbar\Omega_0[(a(\beta)\sinh\theta - \tilde{a}(\beta)\sinh\theta + \tilde{a}^+(\beta)\cosh\theta - a^+(\beta)\cosh\theta)S_-$$

$$+ (a^+(\beta)\sinh\theta - \tilde{a}^+(\beta)\sinh\theta\tilde{a}(\beta) + \cosh\theta - a(\beta)\cosh\theta)S_+]\cos(k_f\hat{x}). \tag{8}$$

Repeating the same steps as those used in [9] for $T = 0$ we have the temperature-dependent equations of motion for the expectation values of coordinate, momentum, spin, annihilation and creation operators

$$\frac{d\hat{x}}{dt} = \frac{P - \tilde{P}}{m}$$

$$\frac{d\hat{P}}{dt} = \hbar k_f \Omega_0 [(a^+(\beta)\cosh\theta - \tilde{a}^+(\beta)\cosh\theta - a(\beta)\sinh\theta + \tilde{a}(\beta)\sinh\theta)S_-$$

$$+(a(\beta)\cosh\theta + \tilde{a}^+(\beta)\sinh\theta - a^+(\beta)\sinh\theta - \tilde{a}(\beta)\cosh\theta)S_+]\sin k_f\hat{x}$$

$$\frac{dS_+}{dt} = i(\omega_f - \omega_a)S_+ + 2i\Omega_0 S_z[a(\beta)\sinh\theta - \tilde{a}(\beta)\sinh\theta + \tilde{a}^+(\beta)\cosh\theta - \tilde{a}(\beta)\cosh\theta]\cos k_f\hat{x}$$

$$\frac{dS_-}{dt} = -i(\omega_f - \omega_a)S_- - 2i\Omega_0 S_z[a^+(\beta)\sinh\theta - \tilde{a}^+(\beta)\sinh\theta + \tilde{a}(\beta)\cosh\theta - a(\beta)\cosh\theta]\cos k_f\hat{x}]$$

$$\frac{da^+}{dt} = -i\Omega_0(S_-\sinh\theta - S_+\cosh\theta)\cos k_f\hat{x}$$

$$\frac{da}{dt} = -i\Omega_0(S_-\cosh\theta - S_+\sinh\theta)\cos k_f\hat{x} \tag{9}$$

$$\frac{d\tilde{a}}{dt} = -i\Omega_0(S_+\sinh\theta - S_-\cosh\theta)\cos k_f\hat{x}$$

$$\frac{\tilde{a^+}}{dt} = i\Omega_0(S_+\cosh\theta - S_-\sinh\theta)\cos k_f\hat{x}$$

$$\frac{dS_z}{dt} = i\Omega_0(B_1 S_- - B_2 S_+)\cos k_f\hat{x},$$

where

$$B_1 = a(\beta)\sinh\theta - \tilde{a}(\beta)\sinh\theta + \tilde{a}^+(\beta)\cosh\theta - a^+(\beta)\cosh\theta,$$

$$B_2 = a^+(\beta)\sinh\theta - \tilde{a}^+(\beta)\sinh\theta + \tilde{a}(\beta)\cosh\theta - a(\beta)\cosh\theta. \tag{10}$$

These equations describe the time evolution of the dynamical variables(expectation values of the operators) in the presence of coupling to a thermal bath with constant temperature, T. We note that interaction of the spin degrees of freedom with a thermal bath is not taken into consideration in these equations.

V. RESULTS AND DISCUSSION

We have solved numerically the system of equations (9) and plotted Poincare surfaces of section (PSS) at various values of temperature. In Fig. 1 the Poincare surface section are plotted for: $\beta = 2$ (a); $\beta = 6$ (b); $\beta = 10$ (c); $\beta = 12$ (d).

It is clear from these plots that for high temperatures the dynamics is fully chaotic, while by decreasing T transition to mixed and regular regime of motion can be observed. To compare the approach for $T = 0$ and our approach for finite T in Fig. 2, PSS are plotted by solving the set of equations (9) and (10) ($T \approx 0$). For T about zero our results for PSS agree with those by Prants [9].

Following the prescription developed in [9] for exploring of instabilities in the QED cavity, another characteristics of chaoticity, so-called Levy flights [23–25] is analyzed. As is well known, chaotic motion in classical system have several quantitative and qualitative characteristics, such as phase-space trajectories, Lyapunov exponent and Levy flights [24, 25]. The latter are the pieces that appear in the trajectory of a particle in a transition from a regular to a chaotic regime of motion. In other words, Levy flights are the chaotic pieces interrupting regular behaviour of the trajectory of an oscillating or regularly moving particle [9, 24]. In Fig. 3, Levy flights for various values of β are plotted. It is clear from this figure that for (a) $T = 0$ and (b) $\beta = 100$ the plots are similar, with the same number of flights. However, by increasing the temperature ($\beta = 10$ and $\beta = 5$) leads to increasing number of flights.

To make our treatment more comprehensive we should consider also the behaviour of the maximum Lyapunov exponent at different temperatures. The maximum Lyapunov exponent characterizes the mean rate of the exponential divergence of initially close trajectories and serves as a quantitative degree of deterministic chaos in the system. In Fig. 4 the maximum Lyapunov exponent is plotted as a function of detuning parameter, δ at different values of temperature. Again, one can observe "more chaos" in the case of finite temperature compared to $\beta = 100$ case. For higher temperatures (smaller β) the Lyapunov exponent becomes more higher than that for lower temperatures.

In Fig. 5 the maximum Lyapunov exponent versus atom field detuning δ and initial atomic momentum p_0 is plotted for $T = 0$ and $\beta = 0.5$. This plot also shows that increasing of the heat bath temperature leads to increasing of maximum Lyapunov exponent for all the values of δ and p_0. However, some "islands" near $= 0$ still exist in this plot. This means that Lyapunov exponent remains as small near $\delta = 0$ at $T \neq 0$.

In all cases(including the case of $T \neq 0$) one can observe that λ becomes equal to zero at $\delta = 0$ that means becoming of our system integrable for $\delta = 0$.

Finally, Fig.6 presents maximum Lyapunov exponent versus δ and β. Again increasing of λ for higher T (smaller β) can be observed. Therefore besides the control parameters α and δ, in the case of finite temperature we have an additional parameter for controlling the dynamics of the atom in a cavity temperature, T.

VI. CONCLUSION

Thus we have studied finite-temperature nonlinear dynamics of an atom coupled to a single mode of the cavity field. Applying the formalism of a real-time finite-temperature field theory to the Jaynes-Cummings Hamiltonian and using the same approach as that used in [9] we have studied classical dynamics of the "atom+cavity mode" system coupling to a thermal bath. The equations of motions for the classical dynamics are obtained by "mapping" of quantum dynamics onto classical one as in the ref. [9].

Using the temperature-dependent equations of motion, dependance of the dynamics on heat-bath effects or finite temperature effects are considered. The results show that the dynamics is quite sensitive to the small changes of temperature. Qualitative characteristics of chaoticity of the system are explored for different values of β. In particular, projections of the Poincare surface section are plotted for different values of temperature. It can be seen by comparing the sensitivity of Poincare surface sections, Levy flights and the maximum Lyapunov exponent to the changes of β with those of δ that the dynamics is more sensitive to the changes of temperature than that of δ. This implies that the temperature of a thermal bath can be considered as an additional control parameter for the dynamics of an atom coupling to a cavity mode.

VII. ACKNOWLEDGMENTS

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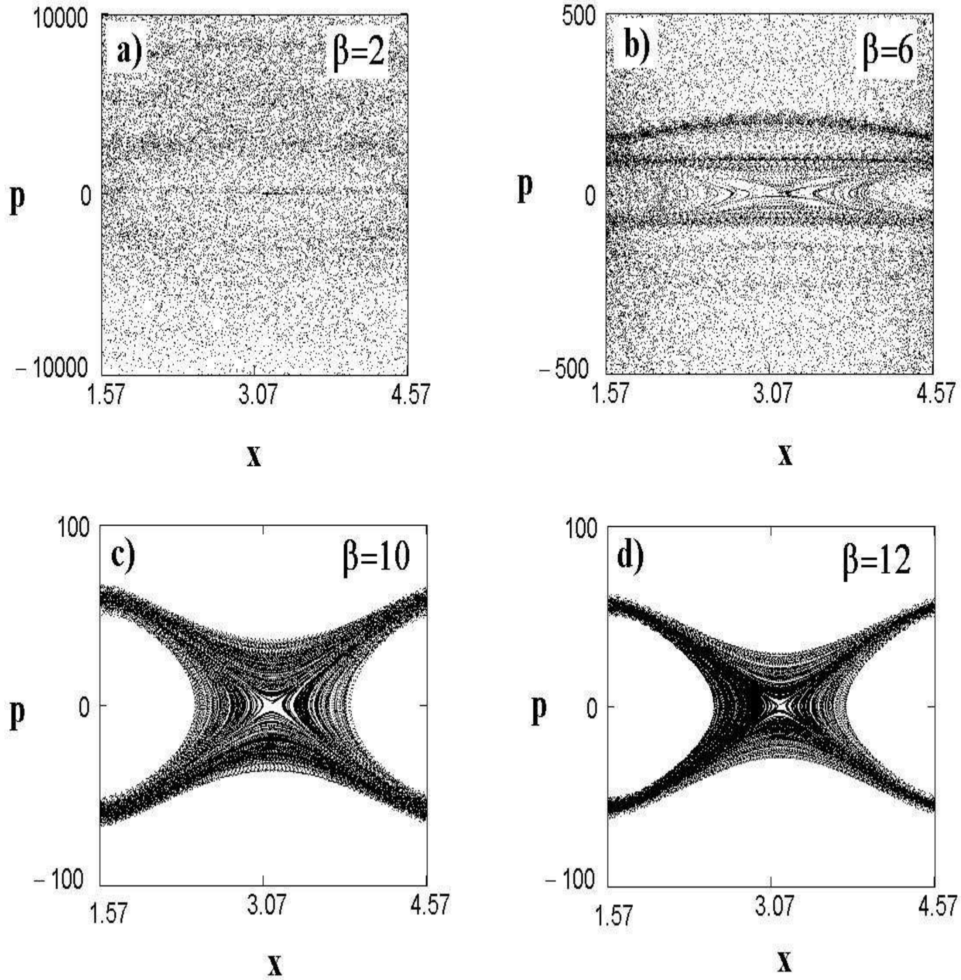

FIG. 1. Projection of the Poincare sections at finite temperature on the plane of the atomic momentum p in units $\hbar k_f$ and the position in units of k_f^{-1}. (a) $\beta = 2$; (b) $\beta = 6$; (c) $\beta = 10$; (d) $\beta = 12$. $\delta = 1.92$ and $s_z(0) = -0.863$ in all cases. x, p are dimensionless.

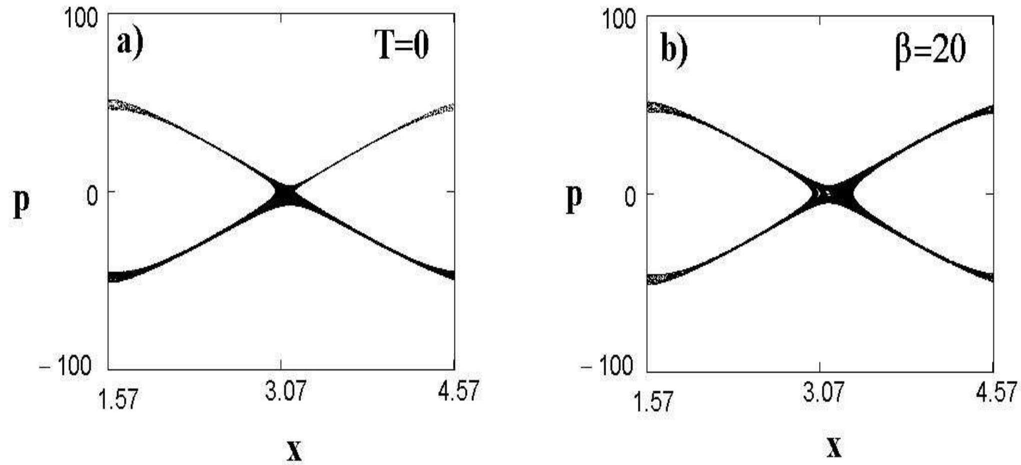

FIG. 2. Projection of the Poincare sections at finite (b) and zero (a) temperature on the plane of the atomic momentum p in units $\hbar k_f$ and the position in units of k_f^{-1}. (a) $T = 0$; (b) $\beta = 20$. $\delta = 1.92$ and $s_z(0) = -0.8660254$ in both cases. x, p are dimensionless.

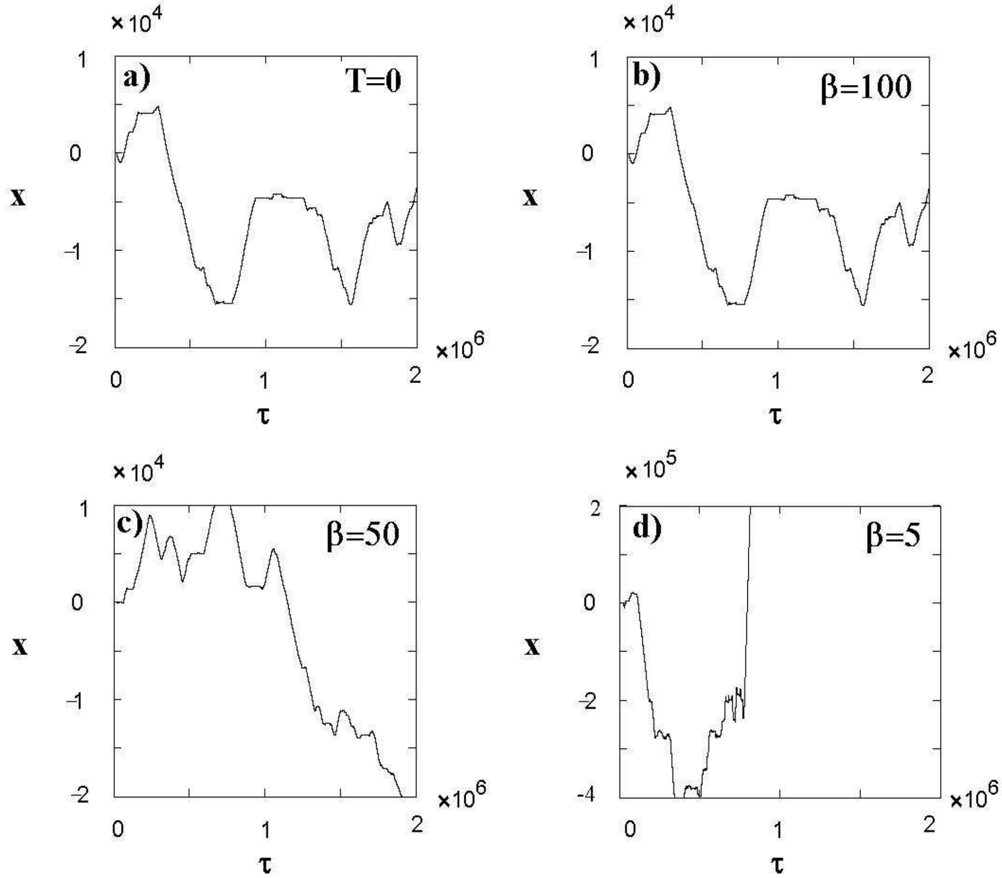

FIG. 3. Levy flights of an atom in a cavity at zero (a) and finite (b,c,d) temperatures. a) $T = 0$; b) $\beta = 100$; c) $\beta = 50$; d) $\beta = 5$. $\delta = 1.2$ and $s_z(0) = -0.8660254$ in all cases. Time is in units of Ω_0^{-1}. x, p, τ are dimensionless.

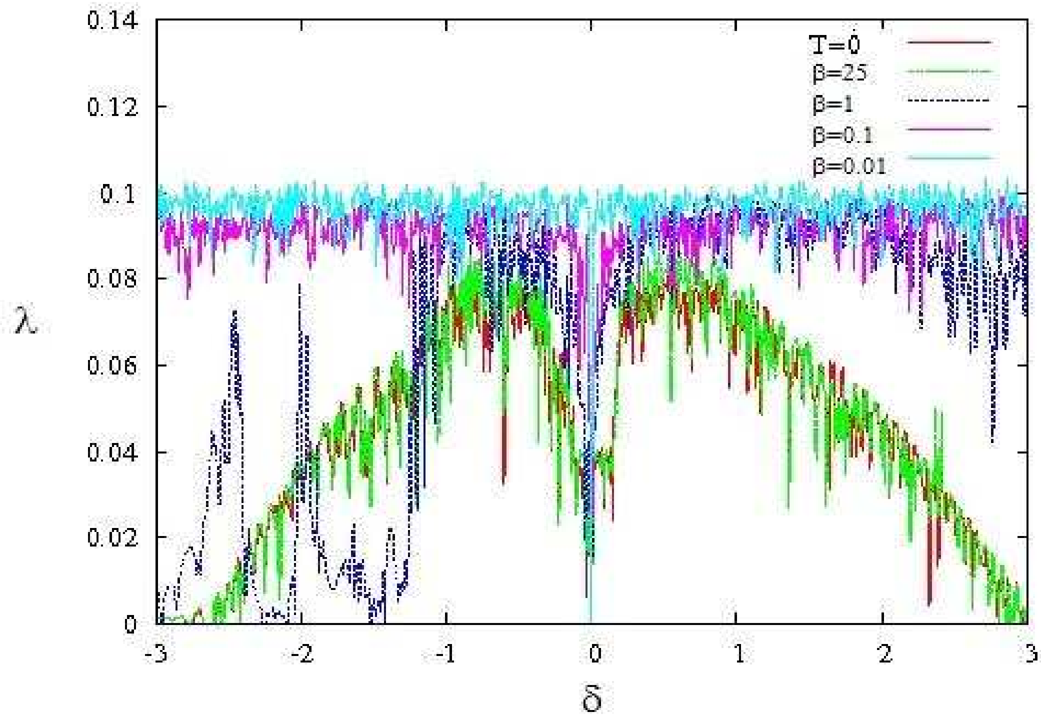

FIG. 4. (Color online) The maximum Lyapunov exponent λ in units of maximal atom-field coupling rate Ω_0 versus the atom-field detuning δ in units Ω_0 at zero and finite temperatures: $\beta = 25$; $\beta = 1$; $\beta = 0.1$; $\beta = 0.01$ and $s_z(0) = 0$ in all cases.

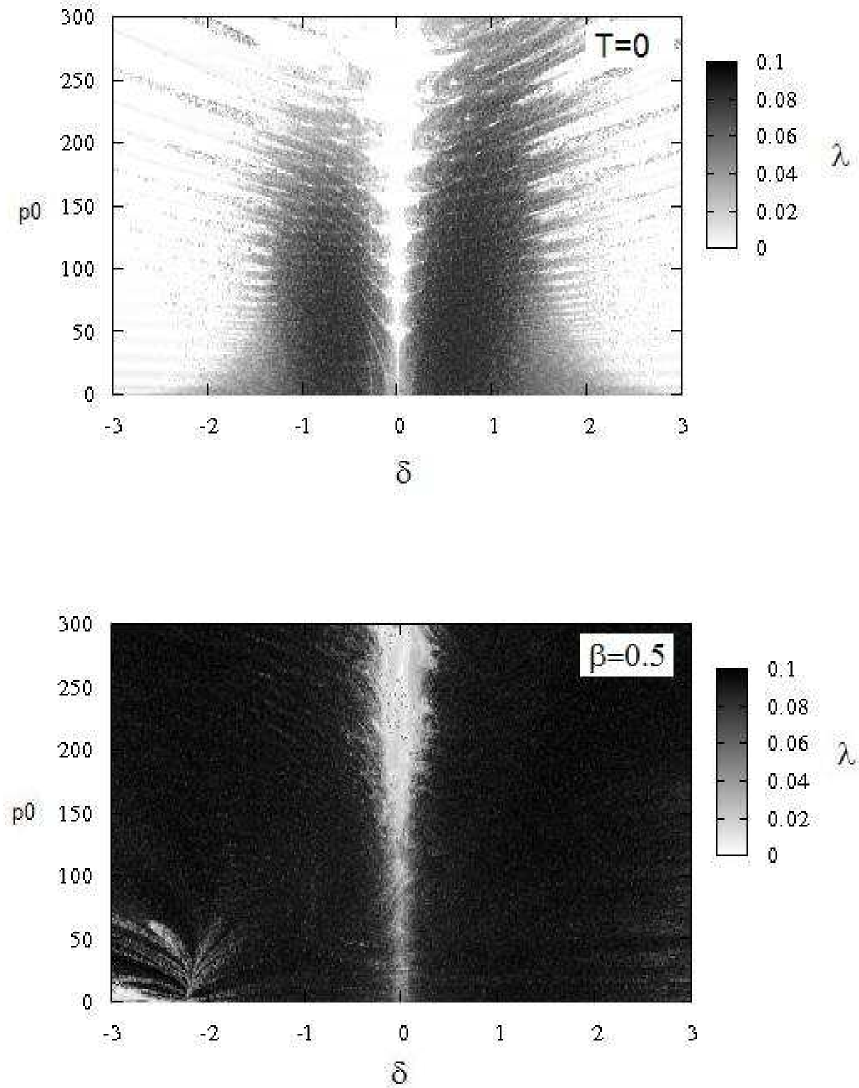

FIG. 5. The maximum Lyapunov exponent λ (in units of maximal atom-field coupling rate Ω_0) versus the atom-field detuning δ (in units Ω_0) at different temperatures: $T = 0$; and $\beta = 0.5$; ($s_z(0) = 0$ in both cases).

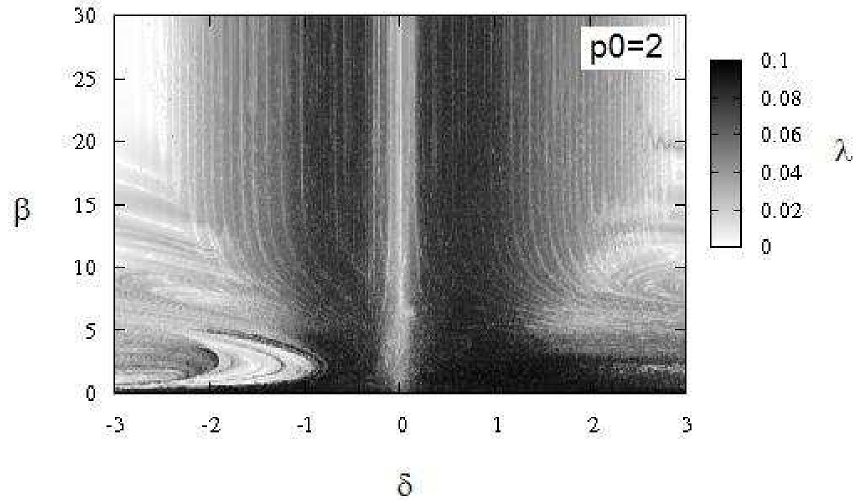

FIG. 6. The maximum Lyapunov exponent λ (in units of maximal atom-field coupling rate Ω_0) versus the atom-field detuning δ (in units Ω_0) and β; ($p_0 = 2$ and $s_z(0) = 0$).